Exploring Fixed Point Theorems in Complex Valued Fuzzy Metric Spaces

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Abstract

The scope of this note encompasses an extension of common fixed-point results from their initial establishment in complex-valued metric spaces to the realm of complex-valued fuzzy metric spaces. By adopting a nuanced approach, these outcomes are further demonstrated within the context of complex-valued fuzzy metric spaces, focusing on employing more lenient contractive criteria. Building upon the groundwork laid by previous researchers and considering a range of other relevant studies, our current research makes significant strides in enhancing and expanding upon their findings. While these prior works have primarily focused on complex-valued metric spaces, our study takes a significant leap forward by applying their insights to complex-valued fuzzy metric spaces. This extension of previous research deepens our understanding of these intricate spaces and opens up new avenues for exploring the complexities inherent in such settings. To enhance the clarity of our core findings, we offer illustrative examples that lend visual support to our assertions.

Keywords

Common fixed point; metric spaces; complex-valued metric spaces(CVMS); complex-valued fuzzy metric spaces (CVFMS)

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Introduction

The foundation of fuzzy set theory was laid by Zadeh [1] in 1965, marking the inception of a concept that has since undergone extensive development by numerous scholars [2-4]. Among these contributions, Kramosil and Michalik [5] notably introduced the innovative notion of fuzzy metric spaces, adding a significant layer of complexity to the evolving landscape of this field. Subsequent refinements to the theory were introduced by George and Veeramani [6], while Grabiec [7] delved into an in-depth exploration of fixed-point theory within the realm of fuzzy metric spaces. Results proved by the authors in [5-7] were basic, conceptual, and important in fixed point theory under metric and fuzzy metric spaces. Many researchers (see [8-12]) have proved results on common fixed points using the theory given in [5-7] in the setting of metric spaces, F- metric spaces, S- metric spaces, and MIFM- spaces.

The inception of CVMS can be attributed to the pioneering work of Azam and his collaborators [13]. Building upon this foundation, Verma and his associates [14] have more recently contributed to the field by introducing significant advancements, such as the 'max' functions, a partial order relation, and the delineation of properties (E-A) and CLRg tailored for complex numbers. These innovative developments have played a crucial role in substantiating fixedpoint theorems within the domain of CVMS. Pioneering the concept of CVFMS, Singh, and team [15], Sharma and Sharma [16] embarked on a groundbreaking endeavor, formulating intricate adaptations of diverse metric space outcomes within this novel framework.

In a recent exploration, H.M. Shrivastava and collaborators [17], investigated a fuzzy version of a system of general random differential equations. The findings from this investigation offer valuable insights for modeling dynamical systems, particularly in addressing environmental challenges such as air pollution. Recently, Z. Eidinejad, R. Saadati, and H.M. Shrivastava [18] approximated a Cauchy additive mapping within a fuzzy Banach space (FBS) by applying a new class of fuzzy control functions. Furthermore, P. Debnath and H.M. Shrivastava [19], extended Kannan's fixed point theorem to the realm of multivalued maps, employing Wardowski's F-contraction. The methodology and results of their study are intriguing and hold the potential for addressing various types of fuzzy equations and solving integral equations in the future. Most recently, P.K. Sharma et al. [20] applied fuzzy fixed point theory to dynamical systems.

In the field of CVFMS, the measurement of distances between points is characterized by using complex-valued fuzzy numbers, diverging from the conventional employment of real numbers. Complex-valued fuzzy metric spaces have applications in various fields. Some notable applications include Quantum Mechanics, Image Processing, Control Systems, Machine Learning, Robotics, etc. These applications highlight the versatility of complex-valued fuzzy metric spaces in dealing with complex data types and uncertainty, making them a valuable tool in various scientific and engineering disciplines. Investigating common fixed points within CVFMS yields valuable insights into the functional dynamics operative within these spaces. This line of inquiry holds significant relevance across diverse domains, including engineering and physics, where the employment of complex-valued functions for modeling physical systems underscores the practical utility of such findings.

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Our present study builds upon the foundational work established by prior researchers, including the contributions of S. Ali [21] and R.K. Verma et al. [14]. Furthermore, we have considered a wide spectrum of pertinent studies in the field. In doing so, our current research extends and significantly advances their findings. This paper aims to extend and illustrate established metric-space outcomes in the CVFMS context, thereby enhancing these results' generalizability.

Preliminaries

Definition 2.1.[13]. Let $\omega_1, \omega_2 \in \mathbb{C}$, where $\omega = \mu + i\nu$. Then, a partial order relation '$\precsim$ ' is established on $\mathbb{C}$ in the following manner:

$\omega_1 \precsim \omega_2 \Leftrightarrow \mathrm{Re}(\omega_1) \leq \mathrm{Re}(\omega_2)$ and $\mathrm{Im}(\omega_1) \leq \mathrm{Im}(\omega_2)$

Hence $\omega_1 \precsim \omega_2$ if one of the following gets satisfied.

(PO1) $\mathrm{Re}(\omega_1) = \mathrm{Re}(\omega_2)$ and $\mathrm{Im}(\omega_1) = \mathrm{Im}(\omega_2)$

(PO2) $\mathrm{Re}(\omega_1) < \mathrm{Re}(\omega_2)$ and $\mathrm{Im}(\omega_1) = \mathrm{Im}(\omega_2)$

(PO3) $\mathrm{Re}(\omega_1) = \mathrm{Re}(\omega_2)$ and $\mathrm{Im}(\omega_1) < \mathrm{Im}(\omega_2)$

(PO4) $\mathrm{Re}(\omega_1) < \mathrm{Re}(\omega_2)$ and $\mathrm{Im}(\omega_1) < \mathrm{Im}(\omega_2)$

In particular, $\omega_1 \lnsim \omega_2$ if $\omega_1 \neq \omega_2$ and one of (PO2), (PO3), and (PO4) gets satisfied, and $\omega_1 \prec \omega_2$ if only (PO4) is satisfied.

It can be noted that;

$0 \precsim \omega_1 \lnsim \omega_2 \Rightarrow |\omega_1| < |\omega_2|,\ \ \omega_1 \precsim \omega_2,\ \omega_2 \prec \omega_3 \Rightarrow \omega_1 \prec \omega_3$.

Definition 2.2.[13]. Let $X \neq \emptyset$. Assume that $d: X \times X \to \mathbb{C}$ satisfies:

(CV1) $0 \precsim d(\mathfrak{l}, \mathfrak{m})$, for all $\mathfrak{l}, \mathfrak{m} \in X$ and $d(\mathfrak{l}, \mathfrak{m}) = 0$ iff $\mathfrak{l} = \mathfrak{m}$;

(CV2) $d(\mathfrak{l}, \mathfrak{m}) = d(\mathfrak{m}, \mathfrak{l})$, for all $\mathfrak{l}, \mathfrak{m} \in X$;

(CV3) $d(\mathfrak{l}, \mathfrak{n}) \precsim d(\mathfrak{l}, \mathfrak{m}) + d(\mathfrak{m}, \mathfrak{n})$, for all $\mathfrak{l}, \mathfrak{m}, \mathfrak{n} \in X$

Then(X, d) is called a CVMS.

Definition 2.3.[14]. The maximum function with partial order relation (POR) '$\precsim$' is defined as

1. $\max\{\mathfrak{p}, \mathfrak{q}\} = \mathfrak{q} \Leftrightarrow \mathfrak{p} \precsim \mathfrak{q}$
2. $\mathfrak{p} \precsim \max\{\mathfrak{q}, \mathfrak{r}\} \Rightarrow \mathfrak{p} \precsim \mathfrak{q}$ or $\mathfrak{p} \precsim \mathfrak{r}$
3. $\mathfrak{p} \precsim \max\{\mathfrak{p}, \mathfrak{q}\} \Rightarrow \mathfrak{p} \precsim \mathfrak{q}$

For any $0 \precsim \mathfrak{p}$, $0 \precsim \mathfrak{q}$ we can show $|\max\{\mathfrak{p}, \mathfrak{q}\}| = \max\{|\mathfrak{p}|, |\mathfrak{q}|\}$

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The min functions can be defined as

(1) min $\{\mathfrak{p}, \mathfrak{q}\} = \mathfrak{p} \Leftrightarrow \mathfrak{p} \lesssim \mathfrak{q}$
(2) min $\{\mathfrak{p}, \mathfrak{q}\} \lesssim \mathfrak{r} \Rightarrow \mathfrak{p} \lesssim \mathfrak{r}$ or $\mathfrak{q} \lesssim \mathfrak{r}$
(3) min $\{\mathfrak{p}, \mathfrak{q}\} \lesssim \mathfrak{q} \Rightarrow \mathfrak{p} \lesssim \mathfrak{q}$

For any $0 \lesssim \mathfrak{p}$, $0 \lesssim \mathfrak{q}$ we can show $|\min\{\mathfrak{p}, \mathfrak{q}\}| = \min\{|\mathfrak{p}|, |\mathfrak{q}|\}$

Example 2.1. let us illustrate the maximum function with the partial order relation (POR) '$\lesssim$' on the set $\mathbb{C}$ of complex numbers using some numerical examples.

Suppose $\mathbb{C} = \{2 - 3i, 4 + 2i, 1 - i, 5 + 4i\}$.We define the partial order relation '$\lesssim$' using the "max" function as follows:

For any two complex numbers x and y in $\mathbb{C}$, $x \lesssim y \Leftrightarrow \max(x, y) = x$. Now, let us apply this relation to pairs of complex numbers:

Is $(2 - 3i) \lesssim (4 + 2i)$?As, $\max(2 - 3i, 4 + 2i) = 4 + 2i$, so $(2 - 3i) \lesssim (4 + 2i)$.

Similarly, we have $(1 - i) \lesssim (5 + 4i)$,$(4 + 2i) \lesssim (5 + 4i)$, and $(2 - 3i) \lesssim (5 + 4i)$.

As we can see from these examples, the "max" function with the partial order relation '$\lesssim$' compares complex numbers based on which one is greater according to the "max" function. If $\max(x, y) = x$, then x is considered $\lesssim y$.

Remark 2.1. To determine whether the "max" function with a partial order relation '$\lesssim$' on the set of complex numbers forms an equivalence relation. We need to consider the properties of an equivalence relation: reflexivity, symmetry, and transitivity. An equivalence relation must be reflexive, which means that for all elements x in the set, $x \lesssim x$ must hold. Using the "max" function, we have $\max(x, x) = x$. This shows that the '$\lesssim$' relation is reflexive because $x \lesssim x$ holds for all complex numbers x. An equivalence relation must be symmetric, meaning that if $x \lesssim y$, $y \lesssim x$ must also hold. In this case, using the "max" function, we have: If $\max(x, y) =x$, then $\max(y, x) = x$. This shows that the '$\lesssim$' relation is symmetric because if $x \lesssim y$, then $y \lesssim x$ also holds. An equivalence relation must be transitive, which means that if $x \lesssim y$ and $y \lesssim z$, then $x \lesssim z$ must be valid. In this case, using the "max" function, we have: If $\max(x, y) = x$ and $\max(y, z) = y$, then $\max(x, z) = x$. This demonstrates the transitivity of the '$\lesssim$' relation, as when $x \lesssim y$ and $y \lesssim z$, it follows that $x \lesssim z$. Thus, the "max" function, is an equivalence relation. Similarly, we can show the "min" function is also an equivalence relation.

Remark 2.2. max $\{\mathfrak{p}, \mathfrak{q}\}$ selects the complex number that is further from the origin in the complex plane. It does not consider the real or imaginary part of the numbers; it solely compares their distances from the origin. In essence, min $\{\mathfrak{p}, \mathfrak{q}\}$selects the complex number closer to the origin in the complex plane, irrespective of their real or imaginary parts. In summary, the 'max' function and 'min' function on the complex plane operate based on the magnitudes (distances from the origin) of the complex numbers and choose the number that is either the maximum or minimum in terms of magnitude, respectively. They do not consider the angles or phases of the complex numbers in the complex plane.

After Zadeh's [1] seminal work on fuzzy set theory, a cadre of scholars [2-4] contributed to this field's core and foundational principles. The pioneer in introducing the theory of fuzzy complex numbers was Buckley [22]. Inspired by Buckley's work, other researchers carried forward the investigation into fuzzy complex numbers. Within this progression, Ramot et al. [23] extended the realm of fuzzy sets to encompass complex fuzzy sets. Drawing inspiration from the groundwork laid by Ramot et al. [23], Singh et al. [15] embarked on an exploration that led to the formulation of CVFMS. They constructed these spaces by employing continuous t-norms as a cornerstone, crafting a Hausdorff topology underpinning their structural integrity. Furthermore, they introduced the innovative concept of Cauchy sequences within the realm of CVFMS, thereby contributing to a deeper understanding of the intricacies within this novel space.

Within the framework of CVFMS, we establish specific results pertaining to fixed points.

Definition 2.4.[23]. The set S (complex fuzzy), defined over a set U (set of the universe of discourse), can be expressed as $S = \{(x, \mu_s(x)): x \in U\}$. Here, the values $\mu_s(x)$ are confined within a unit circle within the complex plane. The function $\mu_s(x)$ is called the membership function and is defined as $\mu_s(x) = r_s(x).\ e^{iw_s(x)}$, where $r_s(x)$ and $w_s(x)$ are real-valued, and $r_s(x) \in [0,1]$.

Definition 2.5. [15]. An operation $* : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, where $\mathbb{C} = \{z = r_s e^{i\theta}: |z| \leq 1, \theta \in [0, \frac{\pi}{2}]\}$, is termed a complex-valued continuous t-norm provided it adheres to the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * e^{i\theta} = e^{i\theta} * a = a, \forall a \in \mathbb{C}$,
(4) $a * b \precsim c * d$ whenever $a \precsim c$ and $b \precsim d$.

Example 2.2. [15]. The following binary operations defined in (i), (ii), and (iii) are complex valued continuous t-norm

(i) $a * b = \min(a, b)$
(ii) $a * b = \max(a + b - e^{i\theta}, 0)$, for a fix $0 \leq \theta \leq \frac{\pi}{2}$.
(iii) $a * b = \begin{cases} \min\{a, b\}, & if\ \max\{a, b\} = e^{i\theta} \\ 0 & otherwise \end{cases}$

Definition 2.6. [15]. $(X, M, *)$ is said to be CVFMS if a complex-valued fuzzy set $M: X \times X \times (0, \infty) \to \mathbb{C}$ (where $X \neq \emptyset$) fulfills the following criteria:

(CF1) $M(a, b, t) > 0$,

(CF2) $M(a, b, t) = e^{i\theta}$ for all $t > 0 \Leftrightarrow a = b$, (CF3) $M(a, b, t) = M(b, a, t)$,

(CF4) $M(a, b, t) * M(b, c, s) \gtrsim M(a, c, t + s)$,

(CF5) $M(a, b, .) : (0, \infty) \to \mathbb{C}$ is continuous, for all a, b, c belong to X, $s, t > 0$.

Singh et al. [15] demonstrated the following results in CVFMS.

Lemma 2.7 [15]. Let $(X, M, *)$ be a CVFMS such that $lim_{t\to\infty}$ M $(a, b, t) = e^{i\theta}$, for all $a, b \in X$, if $M(a, b, kt) \gtrsim M(a, b, t)$, for all $a, b \in X, 0 < k < 1, t \in (0, \infty)$ then $a = b$.
Lemma 2.8 [15]. A sequence $\{b_n\}$ in a CVFMS $(X, M, *)$ with $lim_{t\to\infty}$ $M(a, b, t) = e^{i\theta}$, for all $a, b \in X$ is said to be Cauchy sequence if $\exists\, k$ which lies on $(0, 1)$ such that

$M(b_{n+1}, b_{n+2}, kt) \gtrsim M(b_n, b_{n+1}, t), \forall\, t > 0, n \in I_+$.

Main section/ results

In the last fifty years, fixed point theory has evolved into a captivating and versatile field of study, finding applications in a wide range of areas, including optimization problems, control theory, and differential equations. The fundamental fixed point theorem was originally formulated by Banach [24] in 1922.

The Banach fixed point theorem within CVFMS is reformulated by utilizing the above lemmas, originated by Singh et al. [15].

Theorem 2.1. Let $(X, M, *)$ be a CVFMS with $lim_{t\to\infty}M(a, b, t) = e^{i\theta}, \forall\, a, b \in X, t > 0$. Let T be a self-map on X that satisfies $M(Ta, Tb, kt) \gtrsim M(a, b, t), k \in (0,1)$. Then, T possesses a unique fixed point**.**
Proof:

Let $a_0 \in X$, and sequence $\{a_n\}$ in X defined as $a_n = Ta_{n-1}, n \in \mathbb{N}$

As, $M(a, b, kt) \gtrsim M(a, b, t), k \in (0,1)$. Setting $a = a_{n-1}, b = a_n$, we have

$$M(an-1, an, t) = M(Tan-2, Tan-1, t) \gtrsim M\left(an-2, an-1, \tfrac{t}{k}\right)$$

This implies that $M(a_{n-1}, a_n, t) \gtrsim M\left(a_{n-2}, a_{n-1}, \frac{t}{k}\right)$

Hence, $\{a_n\}$ is a Cauchy sequence in X.

Now we show this sequence converges to $a \in X$.

$$M(a_n, a, kt) \gtrsim M(a_n, a, t)$$

$$M(a_n, a, t) \gtrsim M\left(a_n, a, \tfrac{t}{k}\right) \gtrsim M\left(a_n, a, \tfrac{t}{k^2}\right) \gtrsim \cdots \gtrsim M\left(a_n, a, \tfrac{t}{k^n}\right)$$

Taking $n \to \infty$, we have $lim_{n\to\infty} M(a_n, a, t) \gtrsim e^{i\theta}$

This implies that $a_n \to a$ when $n \to \infty$.

Hence, the sequence $\{a_n\}$ converges to a in X.

Consider, $M(Ta, a, t) \gtrsim M\,(Ta, a_{n+1}, \frac{t}{2}) * M\,(a_{n+1}, a, \frac{t}{2})$

$$= M\,(Ta, Ta_n, \tfrac{t}{2}) * M\,(a_{n+1}, a, \tfrac{t}{2})$$

$$\gtrsim M\,(a, a_n, \tfrac{t}{2k}) * M\,(a_{n+1}, a, \tfrac{t}{2})$$

Taking $n \to \infty$, we get $M(Ta, a, t) \gtrsim M\,(a, a, \frac{t}{2k}) * M\,(a, a, \frac{t}{2})$)

This implies that $M(Ta, a, t) \gtrsim e^{i\theta} * e^{i\theta}$ (Or)

$M(Ta, a, t) = e^{i\theta}$

This implies that $Ta = a$.

Thus, a is a fixed point of T.

Uniqueness: suppose T has another fixed point $b \in X$ and $a \neq b$.

So, $Ta = a$, and $Tb = b$.

Now, $e^{i\theta} \gtrsim M(a, b, t) = M(Ta, T\,b, t) \gtrsim M\,(a, b, \frac{t}{k})$

$$\gtrsim M\,(a, b, \tfrac{t}{k^2}) \gtrsim \cdots \gtrsim M\,(a, b, \tfrac{t}{k^n})$$

This implies that, $e^{i\theta} \gtrsim M\,(a, b, \frac{t}{k^n})$, where $n \in I_+$ and $k \in (0,1)$.

Taking $n \to \infty$, we have $a = b$.

Thus, the fixed point is unique.

Ali [21] formulated the following theorem within the context of a CVMS, encompassing three distinct mappings.

Theorem A [21]. "Let (X, d) be a CVMS and $S, T: X \to X$ be weakly compatible such that

(i) S and T satisfy $(CLRs)$ property and

(ii) $d(Tx, Ty) \precsim \alpha\, d(Sx, Sy) + \frac{\beta d(Tx,Sy) d(Ty,Sx)}{1+d(Sx,Sy)}$ for all $x, y \in X$ and α, β are nonnegative reals with $\alpha + \beta < 1$."

Pathak et al. [14] introduced the following theorems within the framework of a CVMS, involving four distinct mappings.

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Theorem B [14]. "Let (X, d) be a CVMS and $A, B, S, T: X \rightarrow X$ be four self-mappings satisfying:

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$
(ii) $d(Ax, By) \precsim k \max\{d(Sx, Ty), d(By, Sx), d(By, Ty)\}\ \forall\ x, y \in X$, $0 < k < 1$
(iii) The pairs (A, S) and (B, T) are weakly compatible,
(iv) One of the pairs (A, S) or (B, T) satisfies the property (E.A.).

If the range of one of the mappings $S(X)$ or $T(X)$ is a complete subspace of X, then mappings A, B, S, and T have a unique common fixed point in X."

Theorem C [14]. "Let (X, d) be a CVMS and $A, B, S, T: X \rightarrow X$ be four self-mappings satisfying:

(i) $A(X) \subseteq T(X)$,
(ii) $d(Ax, By) \precsim k \max\{d(Sx, Ty), d(By, Sx), d(By, Ty)\}\ \forall\ x, y \in X$, $0 < k < 1$
(iii) The pairs (A, S) and (B, T) are weakly compatible,

If the pairs (A, S) satisfy the (CLRa) property or the pair(B, T) satisfy the (CLRb) property, then mappings A, B, S, and T have a unique common fixed point in X.

If the range of one of the mappings $S(X)$ or $T(X)$ is a complete subspace of X, then mappings A, B, S, and T have a unique common fixed point in X."

We expand the previously presented theorems A, B, and C/results into the realm of CVFMS as outlined below.

Theorem -3.1. Let $(\chi, M, *)$ be a CVFMS with $lim_{t\rightarrow\infty} M(\lambda, \mu, t) = e^{i\theta}$, $\forall\ \lambda, \mu \in \chi$, $t \in (0, \infty)$ and $A, B, S, T : \chi \rightarrow \chi$ be mappings satisfying:

(i) $A(\chi) \subseteq T(\chi)$, $B(\chi) \subseteq S(\chi)$
(ii) $M(A\lambda, B\mu, kt) \succsim \min\{M(S\lambda, T\mu, t), M(B\mu, S\lambda, t), M(B\mu, T\mu, t)\}$, for $0 < k < 1$
(iii) (A, S) and (B, T) are weakly compatible,
(iv) Either (A, S) or (B, T) fulfills the condition (E.A.).

If either the range of S (χ) or T (χ) encompasses a complete subspace of χ, then A, B, S, and T possess a unique fixed point within χ.

Theorem -3.2. Let $(\chi, M, *)$ be a CVFMS with $lim_{t\rightarrow\infty} M(\lambda, \mu, t) = e^{i\theta}$, $\forall\ \lambda, \mu \in \chi$, $t \in (0, \infty)$ and $A, B, S, T : \chi \rightarrow \chi$ be mappings satisfying:

(v) $A(\chi) \subseteq T(\chi)$ or $B(\chi) \subseteq S(\chi)$
(vi) $M(A\lambda, B\mu, kt) \succsim \min\{M(S\lambda, T\mu, t), M(B\mu, S\lambda, t), M(B\mu, T\mu, t)\}$, for $0 < k < 1$
(vii) (A, S) and (B, T) are weakly compatible,

If either the pair (A, S) meets the CLR_A property or the pair(B, T) adheres to the CLR_B property, then mappings A, B, S, and T possess a unique common fixed point within χ.

Proof (Theorem 3.1):

As (A, S) and (B, T) are weakly compatible, $\exists$ coincidence points $\lambda, \mu \in \chi$ such that $A\lambda = S\lambda$ implies $AS\lambda = SA\lambda$, and $B\mu = T\mu$ implies $BT\mu = TB\mu$.

Now by condition (ii)

$M(A\lambda, B\mu, kt) \gtrsim \min\{ M(S\lambda, T\mu, t), M(B\mu, S\lambda, t), M(B\mu, T\mu, t)\}$

$= \min\{ M(A\lambda, B\mu, t), M(B\mu, A\lambda, t), M(B\mu, B\mu, t)\}$

$= \min\{ M(A\lambda, B\mu, t), M(B\mu, A\lambda, t), e^{i\theta}\} = M(A\lambda, B\mu, t)$

Which implies that $A\lambda = B\mu$, and therefore $A\lambda = S\lambda = B\mu = T\mu$ … (1)

Now, let ν be another coincidence point of (A, S) then $A\nu = S\nu$ and so it implies $AS\nu = SA\nu$

Again by (ii) we have

$M(A\nu, B\mu, kt) \gtrsim \min\{ M(S\nu, T\mu, t), M(B\mu, S\nu, t), M(B\mu, T\mu, t)\}$

$= \min\{ M(A\nu, B\mu, t), M(B\mu, A\nu, t), M(B\mu, B\mu, t)\}$

$= \min\{ M(A\nu, B\mu, t), M(B\mu, A\nu, t), e^{i\theta}\} = M(A\nu, B\mu, t)$

Which implies that $A\nu = B\mu$, and therefore $A\nu = S\nu = B\mu = T\mu$ … (2)

By (1) and (2) we have $A\lambda = A\nu$, which shows that (A, S) possesses a unique point of coincidence $\phi = A\lambda = S\lambda$, thus (A, S) possesses a unique fixed point ϕ.

Similarly, we can show the pair (B, T) possesses a unique fixed point.

Suppose the fixed point is $\psi \in \chi$.

Now, by condition (vi)

$M(A\phi, B\psi, kt) = M(\phi, \psi, kt) \gtrsim \min\{ M(S\phi, T\psi, t), M(B\psi, S\phi, t), M(B\psi, T\psi, t)\}$

$\gtrsim \min\{ M(A\phi, B\psi, t), M(B\psi, A\phi, t), M(B\psi, B\psi, t)\}$

$\gtrsim \min\{ M(A\phi, B\psi, t), M(B\psi, A\phi, t), e^{i\theta}\}$

$\gtrsim \min\{ M(\phi, \psi, t), M(\psi, \phi, t), e^{i\theta}\} = M(\phi, \psi, t)$ This

implies that $\phi = \psi$.

Hence A, B, S, and T possess a common fixed point ϕ.

For uniqueness: Let us consider, τ is another common fixed point of A, B, S, and T.

$M(A\phi, B\tau, kt) = M(\phi, \tau, kt) \gtrsim \min\{ M(S\phi, T\tau, t), M(B\tau, S\phi, t), M(B\tau, T\tau, t)\}$

$\gtrsim \min\{ M(A\phi, B\tau, t), M(B\tau, A\phi, t), M(B\tau, B\tau, t)\}$

$\gtrsim \min\{ M(A\phi, B\tau, t), M(B\tau, A\phi, t), e^{i\theta}\}$

$\gtrsim \min\{ M(\phi, \tau, t), M(\tau, \phi, t), e^{i\theta}\} = M(\phi, \tau, t)$ This

implies that $\phi = \tau$.

Hence, A, B, S, and T possess a unique and common fixed point.

Example 3.1.

Let $\chi = R$. Consider the metric $d(\lambda, \mu) = |\lambda - \mu|$. Let $\xi * \zeta = \min\{\xi, \zeta\}$, $\xi, \zeta \in \mathbb{C}$.

$\forall\ t > 0$, and $\lambda, \mu \in \chi$, we define: $M(\lambda, \mu, t) = \frac{e^{i\theta}}{1+\frac{d(\lambda,\mu)}{t}}$

Clearly $(\chi, M,*)$ is CVFMS, and $lim_{t\to\infty} M(\lambda, \mu, t) = e^{i\theta}$

Now, let A, B, S, and T be self-maps, defined as $A\lambda = \frac{2\lambda}{3}$, $B\lambda = \frac{\lambda}{2}$, $S\lambda = 2\lambda$, and $T\lambda = \lambda$.

$$M\ (A\lambda, B\mu, \tfrac{t}{2}) = \frac{\frac{t}{2}e^{i\theta}}{\frac{t}{2}+d(A\lambda, b\mu)}$$

$$M\ (A\lambda, B\mu, \tfrac{t}{2}) = \frac{\frac{t}{2}e^{i\theta}}{\frac{t}{2}+|A\lambda - B\mu|}$$

$$= \frac{\frac{t}{2}e^{i\theta}}{\frac{t}{2}+|2\lambda/3 - \mu/2|}$$

$$= \frac{\frac{t}{2}e^{i\theta}}{\frac{t}{2}+|S\lambda/3 - T\mu/2|}$$

$$= \frac{te^{i\theta}}{t+2|S\lambda/3 - T\mu|}$$

$$\gtrsim \frac{te^{i\theta}}{t+|S\lambda - T\mu|}$$

$$\gtrsim M(S\lambda, T\mu, t)$$

If $\lambda = \mu$ then $M\ (A\lambda, B\mu, \frac{t}{2}) = M(S\lambda, T\mu, t)$

Hence, $M(A\lambda, B\mu, kt) \gtrsim \min\{ M(S\lambda, T\mu, t), M(B\mu, S\lambda, t), M(B\mu, T\mu, t)\}$ for all $\lambda, \mu \in \chi$, $t \in (0, \infty)$ and $0 < k < 1$

Therefore, the maps A, B, S, and T satisfy the condition (ii) of theorem 3.1 for k = ½.

Moreover, 0 is the common fixed point of the maps. Consequently, all the requirements outlined in theorem 3.1 are met.

Proof (Theorem 3.2): Suppose (B, T) satisfies the (CLR_B) property, then $\exists\ \{x_n\}$ in χ such that $lim_{n\to\infty}\ Bx_n = lim_{n\to\infty}\ Tx_n = B\lambda$ for some $\lambda \in \chi$. As $B(\chi) \subseteq S(\chi)$, we have $B\lambda = S\tau$ for some $\tau \in \chi$.

We claim that $A\tau = S\tau$ $(= \rho)$ say. Suppose it is not true then by condition (vi) we have:

$M(A\tau, Bx_n, kt) \gtrsim \min\{ M(S\tau, Tx_n, t), M(Bx_n, S\tau, t), M(Bx_n, Tx_n, t)\}$ $\forall\ \tau, x_n \in \chi, t \in (0, \infty)$ and $0 < k < 1$

$M(A\tau, B\lambda, kt) \gtrsim \min\{ M(B\lambda, B\lambda, t), M(B\lambda, B\lambda, t), M(B\lambda, B\lambda, t)\}$

$M(A\tau, B\lambda, kt) \gtrsim \min\{ e^{i\theta}, e^{i\theta}, e^{i\theta}\}$

$M(A\tau, B\lambda, kt) \gtrsim e^{i\theta}$

Which is a contradiction, thus $A\tau = S\tau$ $(= \rho)$. Hence, $A\tau = S\tau = B\lambda = \rho$.

Thus, τ is a coincidence point of $(A$, S).

As given (A, S) is weak compatible, so $AS\tau = SA\tau = A\rho = S\rho$.

As given $A(\chi) \subseteq T(\chi)$, so $\exists\ \psi \in \chi$ such that $A\tau = T\psi$

Now, we claim that $B\psi = \rho$.

$M(A\lambda, B\mu, kt) \gtrsim \min\{ M(S\lambda, T\mu, t), M(B\mu, S\lambda, t), M(B\mu, T\mu, t)\}$ $\forall\ \lambda, \mu \in \chi, t \in (0, \infty)$ and $0 < k < 1$

$M(A\tau, B\psi, kt) \gtrsim \min\{ M(S\tau, T\psi, t), M(B\psi, S\tau, t), M(B\psi, T\psi, t)\}$

$M(A\tau, B\psi, kt) \gtrsim \min\{ M(\rho, \rho, t), M(B\psi, A\tau, t), M(B\psi, A\tau, t)\}$

$M(A\tau, B\psi, kt) \gtrsim \min\{ e^{i\theta}, M(B\psi, A\tau, t), M(B\psi, A\tau, t)\}$

$M(A\tau, B\psi, kt) \gtrsim M(B\psi, A\tau, t)$

Which implies that $A\tau = B\psi = \rho$

Hence $A\tau = S\tau = \rho = B\psi = T\psi$.

Which shows that ψ is a coincidence point of $(B$, T).

As (B, T) is weak compatible, so $BT\psi = TB\psi = B\rho = T\rho$. Thus, ρ is a common coincidence point of (A, S) and (B, T).

Now consider ρ as a unique and common fixed point of A, B, S, and T.

$M(A\tau, B\rho, kt) \gtrsim \min\{ M(S\tau, T\rho, t), M(B\rho, S\tau, t), M(B\rho, T\rho, t)\}$

$M(\rho, B\rho, kt) \gtrsim \min\{ M(\rho, B\rho, t), M(B\rho, \rho, t), M(B\rho, B\rho, t)\}$

$M(\rho, B\rho, kt) \gtrsim \min\{ M(\rho, B\rho, t), M(B\rho, \rho, t), e^{i\theta}\}$

$M(\rho, B\rho, kt) \gtrsim M(\rho, B\rho, t)$ Which

implies that $\rho = B\rho$.

Thus, $A\rho = B\rho = S\rho = T\rho = \rho$.

Hence, A, B, S, and T possess a common fixed point ρ. The distinctiveness of the fixed point can be readily deduced.

Conclusion

This study revolves around extending conclusions drawn from complete metric spaces (MS) to CVFMS, a process that has been demonstrated to yield accurate outcomes. By introducing a novel adaptation of the less stringent contractive condition, we examined the extended rendition of the outcome. A complex-valued fuzzy version of Banach contraction principles has been proved. Bolstering our arguments, we have included an illustrative example that substantiates our hypotheses and supports our central breakthrough. Within the context of CVFMS, many comprehensive metric space results can be broadened and visually presented along this trajectory.

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